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Weakly-damped focusing nonlinear Schrödinger equations with Dirichlet control

Türker Özsarı

Department of Mathematics, Doğuş University, İstanbul 34722, Turkey

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ABSTRACT

In this article we consider the weakly damped focusing nonlinear Schrödinger equations on bounded domains at the natural H^1-energy level with Dirichlet control acting on a portion of the boundary. We introduce the dynamic extension method for homogenizing the inhomogeneous boundary input. Then, we construct approximate solutions using monotone operator theory. A hidden trace regularity is proved to control the norm of the solutions in a global sense. This allows the use of compactness techniques by which we prove the existence of weak solutions. Finally, using multiplier techniques, we prove the exponential decay of solutions under the assumption that the boundary control also decays in a similar fashion.

1. Introduction and main results

Assume that Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$ where Γ_0 and Γ_1 are nonempty disjoint connected $(n-1)$-dimensional manifolds.

We consider the initial–boundary value problem,

$$\begin{aligned} &iu_t = \Delta u + F(u), \quad \text{in } Q = \Omega \times (0,T),\\ &u = \begin{cases} 0, & \text{on } \Sigma_0 = \Gamma_0 \times (0,T),\\ h, & \text{on } \Sigma_1 = \Gamma_1 \times (0,T), \end{cases}\\ &u(0) = u_0, \quad \text{in } \Omega. \end{aligned} \tag{1.1}$$

Our focus is to study exponential stabilization of the problem in (1.1) with

$$F(u) = f(u) - iau, \qquad f(u) = |u|^p u, \quad a \geqslant 0,\ p \in \left(0, \frac{4}{n+2}\right). \tag{1.2}$$

Note that if $a = 0$ one has the classical NLS on domains without damping.

The boundary input $h : (0,\infty) \times \partial\Omega \to \mathbb{C}$ belongs to the space

$$H^1(\Sigma_1) \equiv L_2\big(0,T; H^1(\Gamma_1)\big) \cap H^1\big(0,T; L_2(\Gamma_1)\big) \tag{1.3}$$

E-mail address: tozsari@dogus.edu.tr.
URL: http://www3.dogus.edu.tr/tozsari.

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for each $T > 0$, and the initial state

$$u_0 \in H^1_{0,\Gamma_0}(\Omega) = \{\varphi \in H^1(\Omega)\colon \varphi|_{\Gamma_0} = 0\}. \tag{1.4}$$

Moreover, the initial state and boundary control are assumed to satisfy the natural compatibility condition

$$u_0|_{\Gamma_1} = h(0). \tag{1.5}$$

There has been great effort towards the study of NLS posed on $\mathbb{R}^n$ recently. However, in the case of a bounded domain, the nonlinear dynamics of the solutions is generally different than in the case of $\mathbb{R}^n$, see [2] and [3]. The methods used in the case of $\mathbb{R}^n$ such as Strichartz estimates, are not always true or available on domains. Therefore, the corresponding problems on domains are more difficult due to the lack of tools which provide sharp global results.

In the study of NLS with inhomogeneous boundary conditions, serious obstacles arise, one of which is the emergence of boundary integrals involving the directional derivative of the solution. This also spoils the classical conservation laws of mass and energy for NLS, and the methods based on these conservation laws do not work anymore. Therefore, obtaining the corresponding existence and long-time behavior results is nontrivial and the analysis involved is more subtle.

Of particular relevance to our results in this article is [16] where the author studies exponential stabilization of the solutions of (1.1) with homogeneous boundary condition ($h \equiv 0$). We have extended these stabilization results to defocusing NLS with inhomogeneous Dirichlet boundary condition in [11]. In this paper, we continue this program and answer the stabilization problem for the focusing NLS.

The global existence of weak solutions for (1.1) without damping ($a = 0$) at the H^1-level has been recently established in [1] for $n \geqslant 2$ and $0 < p \leqslant \frac{2}{n}$ with a highly regular C^3-boundary input. Also the defocusing case for all dimensions and power of nonlinearities are treated in [15] without damping, and in [11] with damping. It is well known (see e.g. [6]) that when $h \equiv 0$, solutions of (1.1) might blow up if $p \geqslant \frac{4}{n}$. Therefore, the gap $\frac{2}{n} < p < \frac{4}{n}$ is conjectured. In this paper, we partially fill this gap and improve the result in [1] in various directions. First, we show global existence for $0 < p < \frac{4}{n+2}$, which improves $0 < p \leqslant \frac{2}{n}$ for dimensions $n > 2$. Secondly, we obtain this result with boundary inputs taken from comparatively rougher spaces, namely $H^1(\Sigma_1)$ instead of C^3, using the dynamic extension method explained in the next section. Finally, we prove that solutions decay to zero exponentially in the presence of a damping term ($a > 0$) under the assumption that the boundary control also decays in a similar fashion.

Studying the existence and long time behavior of solutions for (1.1) has important implications for other areas of control theory since one can ask the question of exact boundary controllability as a second step. Formally, the exact boundary controllability question is:

Given initial and final states u_0 and u_T, is there a control function h which steers a solution of (1.1) from u_0 to u_T?

There are recent studies on the exact boundary controllability problem [14,13,5,17]. However the results obtained so far have local character, and the problem is still open in the general sense even for nice nonlinearities, e.g., Lipschitz.

There are various techniques for studying the controllability problems for nonlinear PDEs. In Remark 2.2, we reduce the exact boundary controllability problem to a fixed point formulation by using the dynamic extension method we present in the next section.

Main results

In this paper, we prove the following global existence and stabilization results.

Theorem 1.1 *(Global existence and hidden trace regularity). Let $T > 0$ be arbitrary and $a \geqslant 0$. Then problem (1.1)–(1.5) has a weak H^1-solution u such that*

$$u \in L_\infty\big(0, T; H^1_{0,\Gamma_0}(\Omega)\big), \quad u_t \in L_\infty\big(0, T; H^{-1}(\Omega)\big). \tag{1.6}$$

Moreover, the (hidden) trace regularity

$$\frac{\partial u}{\partial n} \in L_2(\Sigma) \tag{1.7}$$

is satisfied.

Theorem 1.2 *(Exponential stabilization). Suppose that the boundary control satisfies the decay condition described by the integral inequality*

$$\int_0^\infty e^{2as} \|h(s)\|_w^2 \, ds = M < \infty. \tag{1.8}$$

Let $T > 0$ be arbitrary and $a > 0$. Then, (1.1)–(1.5) *has a weak H^1-solution u such that for all $b < a$, there exists a positive constant C which depends on Ω, p, a, $\|u_0\|_{H^1_{0,\Gamma_0}(\Omega)}$, M, and b so that*

$$\|u(t)\|_{H^1_{0,\Gamma_0}(\Omega)} \leqslant Ce^{-bt} \quad \text{for a.e. } t \in [0,T]. \tag{1.9}$$

Notation 1.1. $\|h(t)\|_w = \|h(t)\|^2_{H^1(\Gamma_1)} + \|h_t(t)\|^2_{L_2(\Gamma_1)}$. Since we assume that $\partial\Omega$ is sufficiently smooth $L_{p+2}(\Gamma_1)$ norm of h is imbedded in $H^1(\Gamma_1)$. Otherwise, one also needs to add this to the definition of w-norm.

The proofs of Theorem 1.1 and Theorem 1.2 require a combination of different techniques. The first step is to homogenize the equation via dynamic extension method. Secondly, approximate solutions are constructed for the homogeneous equation using monotone operator theory. Then some special multipliers are introduced to be able to control the trace of the normal derivative of the solutions. Once this is achieved, compactness techniques are used to extract a subsequence of the approximate solutions, which converges to a sought-after weak solution of (1.1) which satisfies the desired properties.

2. Dynamic extension

We begin with a homogenization method which preserves the optimal regularity level of the solutions. The method is comprised of two main steps. The first step is solving the corresponding linear equation with the same initial and boundary values. This is equivalent to defining an extension operator which maps the boundary value to the solution of the corresponding linear equation which is of course defined on the whole domain. The second step is to homogenize the original problem using this extension. We call this method *dynamic extension*.

More precisely, let v be the solution of the following equation.

$$\begin{aligned}
&iv_t = \Delta v, \quad \text{in } Q,\\
&v = \begin{cases} 0, & \text{on } \Sigma_0,\\ h, & \text{on } \Sigma_1,\end{cases}\\
&v(0) = u_0, \quad \text{in } \Omega,
\end{aligned} \tag{2.1}$$

and let's make the definition $w := u - v$. Then w must solve the following homogeneous equation.

$$\begin{aligned}
&iw_t = \Delta w + F_v(w), \quad \text{in } Q,\\
&w = 0, \quad \text{on } \Sigma := \partial\Omega \times (0,T),\\
&w(0) = 0, \quad \text{in } \Omega,
\end{aligned} \tag{2.2}$$

where $F_v(w) = f_v(w) - ia(w+v)$ with $f_v(w) = |w+v|^p(w+v)$.

The linear problem (2.1) is well studied. Indeed, one has the following regularity results.

Theorem 2.1. *(See [8].) Consider the problem* (2.1) *with control function*

$$h \in H^{1,1}(\Sigma_1),$$

and the initial data

$$u_0 \in H^1_{0,\Gamma_0}(\Omega).$$

Suppose also that the compatibility condition $h(0) = u_0|_{\Gamma_1}$ is satisfied. Then (2.1) *admits a unique solution v such that*

$$v \in C\big([0,T]; H^1_{0,\Gamma_0}(\Omega)\big), \quad v_t \in C^1\big([0,T]; H^{-1}(\Omega)\big)$$

together with

$$\frac{\partial v}{\partial n} \in L_2(\Sigma).$$

Moreover, the mapping

$$h, u_0 \to v, v_t, \frac{\partial v}{\partial n}$$

is continuous from

$$H^{1,1}(\Sigma_1) \times H^1_{0,\Gamma_0}(\Omega)$$

to

$$C\big([0,T];H^1_{0,\Gamma_0}(\Omega)\big)\times C^1\big([0,T];H^{-1}(\Omega)\big)\times L_2(\Sigma).$$

Theorem 2.2. *(See [8].) Consider the problem* (2.1) *with control function*

$$h\in L_2(\Sigma_1),$$

and the initial data

$$u_0\in L_2(\Omega).$$

Then (2.1) *admits a unique solution* v *such that*

$$v\in C\big([0,T];L_2(\Omega)\big).$$

Well-posedness of the linear problem in particular implies that the extension process is well-defined.
Now we are ready to introduce the concept of weak solution for (1.1).

Definition 2.1. We say that $u:=v+w$ is a weak H^1-solution of (1.1) if v is the unique solution of (2.1) in the transposition sense and w is a weak H^1_0-solution of (2.2).

In Definition 2.1, w is a weak H^1_0-solution of (2.2) if

$$w\in L_\infty\big(0,T;H^1_0(\Omega)\big)\cap W^{1,\infty}\big(0,T;H^{-1}(\Omega)\big)$$

and w satisfies:

$$(iw,\varphi)_{L_2(\Omega)}+\int_0^t(\nabla w,\nabla\varphi)_{L_2(\Omega)}\,ds-\int_0^t\big[f_v(w),\varphi\big]_{L_{(p+2)'}(\Omega),L_{(p+2)}(\Omega)}\,ds+ia\int_0^t(w+v,\varphi)_{L_2(\Omega)}\,ds=0 \tag{2.3}$$

for all $\varphi\in H^1_0(\Omega)$ and for almost all $t\in[0,T]$.

Notation 2.1. In (2.3), $(\cdot,\cdot)_{L_2(\Omega)}$ denotes the inner product of Hilbert space $L_2(\Omega)$. $(p+2)'=\frac{p+2}{p+1}$, where $'$ denotes the conjugate index. $[\cdot,\cdot]_{X',X}$ denotes a Banach space pairing between a Banach space X and its dual X'.

Representation formulae

Let A be the operator on $L_2(\Omega)$ defined by

$$\begin{cases} D(A)=H^2(\Omega)\cap H^1_0(\Omega),\\ Au=-\Delta u, \quad \forall u\in D(A).\end{cases} \tag{2.4}$$

Then A is positive, self-adjoint, and $\mathcal{A}=iA$ is a skew-adjoint operator generating a group of isometries $(\mathscr{W}(t))_{t\in\mathbb{R}}$ in $H^{-1}(\Omega)$ [4]. Then a solution of (2.2) can be written in the integral form

$$w(t)=-i\int_0^t\mathscr{W}(t-s)F_{v(s)}\big(w(s)\big)\,ds. \tag{2.5}$$

Actually, we have the following representation formula.

Proposition 2.1. *If* $u\in L_\infty(0,T;H^1_{0,\Gamma_0}(\Omega))$*, then* u *is a weak* H^1*-solution of* (1.1) *if and only if*

$$u(t)=v(t)-i\int_0^t\mathscr{W}(t-s)F\big(u(s)\big)\,ds \tag{2.6}$$

for almost all $t\in[0,T]$.

Proof. This follows from the classical Duhamel's formula provided that $F \in C(H^1(\Omega), H^{-1}(\Omega))$ and F is bounded on bounded sets, see for example [4]. Indeed, by Hölder's and Gagliardo–Nirenberg inequalities, it follows that

$$\begin{aligned}\left|\big(F(u),\varphi\big)_{L_2(\Omega)}\right| &= \left|\int_\Omega |u|^p u\bar{\varphi}\,dx - ia\int_\Omega u\bar{\varphi}\,dx\right| \\ &\leqslant \|u\|_{L_{p+2}(\Omega)}^{p+1}\|\varphi\|_{L_{p+2}(\Omega)} + a\|u\|_{L_2(\Omega)}\|\varphi\|_{L_2(\Omega)} \\ &\leqslant C\big(\|u\|_{H^1(\Omega)}^{p+1} + \|u\|_{H^1(\Omega)}\big)\|\varphi\|_{H^1(\Omega)} \end{aligned} \tag{2.7}$$

for each $\varphi \in H_0^1(\Omega)$. Hence, $F(u) \in H^{-1}(\Omega)$ for each $u \in H^1(\Omega)$ and F is bounded on bounded sets. On the other hand,

$$\begin{aligned}\left|\big(F(u_1)-F(u_2),\varphi\big)_{L_2(\Omega)}\right| &\leqslant \int_\Omega \left|\big(|u_1|^p u_1 - |u_2|^p u_2\big)\bar{\varphi}\right|dx + a\int_\Omega \left|(u_1-u_2)\bar{\varphi}\right|dx \\ &\leqslant \big(\|u_1\|_{L_{p+2}(\Omega)}^{p+1} + \|u_2\|_{L_{p+2}(\Omega)}^{p+1}\big)\|u_1-u_2\|_{L_{p+2}(\Omega)}\|\varphi\|_{L_{p+2}(\Omega)} + a\|u_1-u_2\|_{L_2(\Omega)}\|\varphi\|_{L_2(\Omega)} \\ &\leqslant C\big(\|u_1\|_{H^1(\Omega)}^{p+1} + \|u_2\|_{H^1(\Omega)}^{p+1}\big)\|u_1-u_2\|_{H^1(\Omega)}\|\varphi\|_{H^1(\Omega)}. \end{aligned} \tag{2.8}$$

Therefore, F is continuous. □

Remark 2.1. Note that in (2.6), the initial and boundary conditions of (1.1) are satisfied thanks to the dynamic extension term $v(t)$.

Remark 2.2. The representation formula in (2.6) can also be used to formulate the exact boundary controllability problem for NLS. To see this, let u_T be the final state to which we want to steer the solutions from the initial state, say $u_0 \equiv 0$ for simplicity. Suppose that the linear Schrödinger equation in (2.1) is exactly controllable, which is indeed well known in the literature. Therefore, we have an onto map

$$h \to v(T), \tag{2.9}$$

from control space to the state space. This map induces a bijection Υ from a subset of the control space to the state space. Therefore, we can define a control as follows:

$$h = \Upsilon^{-1}\left(u_T + i\int_0^T \mathscr{W}(T-s)F\big(u(s)\big)\,ds\right). \tag{2.10}$$

Now, let Ψ be the dynamic extension operator, i.e.,

$$\big[\Psi(h)\big](t) = v(t).$$

Thus, the right-hand side of (2.6) can be written as an operator

$$\big[\Phi(u)\big](t) = \left[\Psi\left(\Upsilon^{-1}\left(u_T + i\int_0^T \mathscr{W}(T-s)F\big(u(s)\big)\,ds\right)\right)\right](t) - i\int_0^t \mathscr{W}(t-s)F\big(u(s)\big)\,ds. \tag{2.11}$$

Hence, the exact boundary controllability for NLS is equivalent to finding a fixed point of the nonlinear operator Φ. Note that this fixed point satisfies $u(0) = u_0$ and $u(T) = u_T$.

It is difficult to directly obtain well-posedness from (2.6) unless some Strichartz type estimates are proved for general domains, which is a difficult problem. Instead, we will rely on the monotone operator theory, which can be effectively used to approximate m-accretive nonlinearities with Lipschitz maps known as Yosida approximations.

Therefore, let's first consider the following Cauchy problem in the Lipschitz setting.

$$\begin{aligned} &w'(t) = \mathcal{A}w(t) + g\big(v(t) + w(t)\big), \quad t > 0, \\ &w(0) = 0. \end{aligned} \tag{2.12}$$

Then, one has the following lemmas, where the proofs follow from the techniques presented in [12].

Lemma 2.1. *Suppose $v : [0,T] \to L_2(\Omega)$ is continuous in t on $[0,T]$, $g : L_2(\Omega) \to L_2(\Omega)$ is uniformly Lipschitz continuous on $L_2(\Omega)$ with Lipschitz constant $L > 0$. Then (2.12) has a unique solution $w \in C([0,T]; L_2(\Omega))$.*

Proof. Let Φ be the operator on $C([0,T];L_2(\Omega))$ (equipped with the sup norm) defined by

$$\Phi(w)(t) = -i\int_0^t \mathscr{W}(t-s)g\big(v(s)+w(s)\big)\,ds.$$

It follows that

$$\big\|\Phi(w)(t)-\Phi(z)(t)\big\|_{L_2(\Omega)} \leqslant Lt\|w-z\|_\infty.$$

By induction,

$$\big\|\Phi^n(w)(t)-\Phi^n(z)(t)\big\|_{L_2(\Omega)} \leqslant \frac{(Lt)^n}{n!}\|w-z\|_\infty \leqslant \frac{(LT)^n}{n!}\|w-z\|_\infty.$$

Φ^n is a contraction for n large enough. Hence, Φ has a fixed point which is a solution of (2.12) in $C([0,T];L_2(\Omega))$.

To prove uniqueness, let w and z be two continuous solutions to (2.12). Then

$$\|w-z\|_{L_2(\Omega)} \leqslant \int_0^t \big\|\mathscr{W}(t-s)\big(g\big(v(s)+w(s)\big)-g\big(v(s)+z(s)\big)\big)\big\|_{L_2(\Omega)}\,ds \leqslant L\int_0^t \big\|w(s)-z(s)\big\|_{L_2(\Omega)}\,ds. \tag{2.13}$$

By Gronwall's inequality, we obtain $\|w-z\|_{L_2(\Omega)} \leqslant 0$. This implies $w=z$. □

Lemma 2.2. *Suppose $v:[0,T]\to L_2(\Omega)$ is Lipschitz continuous in t on $[0,T]$, $g:L_2(\Omega)\to L_2(\Omega)$ is Lipschitz continuous on $L_2(\Omega)$, and w is the continuous solution of the initial value problem* (2.12) *then w is the strong solution of this initial value problem.*

Proof. Let $\theta\in(0,T-t)$ and M, N be two constants such that $\|g(v(s)+w(s))\|_{L_2(\Omega)} \leqslant M$ and

$$\int_0^t \big\|v(s+\theta)-v(s)\big\|_{L_2(\Omega)}\,ds \leqslant \theta N$$

on $[0,T]$.

Observe that,

$$\begin{aligned} w(t+\theta)-w(t) &= \int_0^{t+\theta} \mathscr{W}(t+\theta-s)g\big(w(s)+v(s)\big)\,ds - \int_0^t \mathscr{W}(t-s)g\big(w(s)+v(s)\big)\,ds \\ &= \int_0^t \mathscr{W}(t-s)\big[g\big(w(s+\theta)+v(s+\theta)\big)-g\big(w(s)+v(s)\big)\big]\,ds \\ &\quad + \int_0^\theta \mathscr{W}(t+\theta-s)g\big(w(s)+v(s)\big)\,ds. \end{aligned} \tag{2.14}$$

Hence,

$$\begin{aligned} \big\|w(t+\theta)-w(t)\big\|_{L_2(\Omega)} &\leqslant \theta M+\theta LN+L\int_0^t \big\|w(s+\theta)-w(s)\big\|_{L_2(\Omega)}\,ds \\ &\leqslant \theta C+L\int_0^t \big\|w(s+\theta)-w(s)\big\|_{L_2(\Omega)}\,ds. \end{aligned} \tag{2.15}$$

We conclude by Gronwall's lemma that

$$\big\|w(t+\theta)-w(t)\big\|_{L_2(\Omega)} \leqslant Ce^{LT}\theta.$$

Now, it follows that the map $s\to g(w(s)+v(s))$ is Lipschitz in time, but then w is a strong solution by the classical semigroup theory. □

3. Approximate solutions

Consider the nonlinear operator $\mathcal{B}$ in $L_2(\Omega)$ defined by

$$\begin{cases} D(\mathcal{B}) = \{u \in L_2(\Omega)\colon\ |u|^p u \in L_2(\Omega)\}, \\ \mathcal{B}u = |u|^p u, \quad \forall u \in D(\mathcal{B}). \end{cases}$$

Then $\mathcal{B}$ satisfies the following operator theoretic property.

Lemma 3.1. *(See [10].) $\mathcal{B} : L_2(\Omega) \supset D(\mathcal{B}) \to L_2(\Omega)$ is an m-accretive operator.*

By m-accretivity of $\mathcal{B}$, we can define its Lipschitz continuous Yosida approximations on $L_2(\Omega)$ by

$$\mathcal{B}_N = N(I - J_N) = \mathcal{B} J_N,$$

where

$$J_N = \left(1 + \frac{1}{N}\mathcal{B}\right)^{-1}$$

are resolvents. We can find ψ and ψ_N such that $\mathcal{B} = \partial\psi$ and $\mathcal{B}_N = \partial\psi_N$, respectively. Indeed,

$$\psi(u) := \begin{cases} \frac{1}{p+2}\|u\|_{L_{p+2}(\Omega)}^{p+2} & \text{for } u \in L^{p+2}(\Omega), \\ \infty & \text{otherwise}, \end{cases} \tag{3.1}$$

and

$$\psi_N(u) := \min_{\phi \in L_2(\Omega)} \left\{\frac{N}{2}\|\phi - u\|^2 + \psi(\phi)\right\} = \frac{1}{2N}\|\mathcal{B}_N u\|^2 + \psi(J_N u) \quad \text{for } u \in L_2(\Omega). \tag{3.2}$$

Before defining the notion of approximate solutions for the nonlinear problem (1.1), we will need the following regularity result for linear Schrödinger equations.

Theorem 3.1. *Consider the problem* (2.1) *with control function*

$$h \in C([0,T]; H^2(\Gamma_1)) \cap C^2([0,T]; L_2(\Gamma_1)),$$

and the initial data

$$u_0 \in H^2(\Omega) \cap H^1_{0,\Gamma_0}(\Omega).$$

Suppose also that the compatibility condition $h(0) = u_0|_{\Gamma_1}$ is satisfied. Then (2.1) *admits a unique solution v such that*

$$v \in C([0,T]; H^2(\Omega) \cap H^1_{0,\Gamma_0}(\Omega)), \quad v_t \in C^1([0,T]; L^2(\Omega)).$$

Proof. The proof of the corresponding problem for wave equations [7] also works for the Schrödinger equation. Indeed, we set

$$z = \frac{d}{dt}v.$$

Then, we have the z-problem:

$$\begin{aligned} &iz_t = \Delta z, \quad \text{in } Q, \\ &z = \begin{cases} 0, & \text{on } \Sigma_0, \\ h_t, & \text{on } \Sigma_1, \end{cases} \\ &z(0) = \Delta u_0, \quad \text{in } \Omega. \end{aligned} \tag{3.3}$$

Then according to Theorem 2.2 we have

$$z = \frac{d}{dt}v \in C([0,T]; L_2(\Omega)).$$

Hence, we have

$$\Delta v = i v_t \in C\big([0,T]; L_2(\Omega)\big),$$
$$v|_\Gamma \in C\big([0,T]; H^{\frac{3}{2}}(\Omega)\big). \tag{3.4}$$

By the well-known regularity theory of the Poisson equation, we have $v \in C([0,T]; H^2(\Omega))$. $\square$

Now, let

$$h_N \in C\big([0,T]; H^2(\Gamma_1)\big) \cap C^2\big([0,T]; L_2(\Gamma_1)\big),$$

and $u_{0N} \in H^2(\Omega) \cap H^1_{0,\Gamma_0}(\Omega)$ be sequences of functions such that $h_N \to h$ strongly in $H^1(\Sigma_1)$ and $u_{0N} \to u_0$ strongly in $H^1_{0,\Gamma_0}(\Omega)$. Then the linear problem (2.1) with the boundary control h_N and the initial state u_{0N} admits a solution v_N for each N such that

$$v_N \in C\big([0,T]; H^2(\Omega) \cap H^1_{0,\Gamma_0}(\Omega)\big), \quad v_{Nt} \in C^1\big([0,T]; L^2(\Omega)\big)$$

by Theorem 3.1. Moreover,

$$v_N, v_{Nt}, \frac{\partial v_N}{\partial n} \to v, v_t, \frac{\partial v}{\partial n}$$

in

$$C\big([0,T]; H^2(\Omega) \cap H^1_{0,\Gamma_0}(\Omega)\big) \times C\big([0,T]; H^{-1}(\Omega)\big) \times L_2(\Sigma)$$

by the well-posedness result in Theorem 2.1. Note that, v_N is Lipschitz as a function from $[0,T]$ to $L_2(\Omega)$.

Now, we can consider the following approximate problems.

$$w_N'(t) = \mathcal{A} w_N(t) + g_N\big(v_N(t) + w_N(t)\big), \quad t > 0,$$
$$w_N(0) = 0 \tag{3.5}$$

where

$$g_N = -i\mathcal{B}_N - aI.$$

By Lemma 2.1 and Lemma 2.2, (3.5) has a solution w_N such that

$$w_N \in C\big([0,T]; H^2(\Omega) \cap H^1_0(\Omega)\big), \quad w_{Nt} \in C^1\big([0,T]; L^2(\Omega)\big)$$

for each N. Hence, we are ready to define approximate solutions of (1.1) by

$$u_N = v_N + w_N.$$

Note that now u_N satisfies the following approximate problem on which we can run the machinery of multipliers.

$$i u_{Nt} = \Delta u_N + i g_N(u_N), \quad \text{in } Q,$$
$$u_N = \begin{cases} 0, & \text{on } \Sigma_0, \\ h_N, & \text{on } \Sigma_1, \end{cases}$$
$$u_N(0) = u_{N0}, \quad \text{in } \Omega. \tag{3.6}$$

The solution of (3.6) satisfies the following mass and energy identities.

$$\frac{d}{dt}\frac{1}{2}\|u_N\|^2_{L_2(\Omega)} = -a\|u_N\|^2_{L_2(\Omega)} + \operatorname{Im}\int_{\Gamma_1} \frac{\partial u_N}{\partial n}\bar{h}_N \, d\Gamma, \tag{3.7}$$

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u_N\|^2_{L_2(\Omega)} - \psi_N(u_N)\right) = -2a\left(\frac{1}{2}\|\nabla u_N\|^2_{L_2(\Omega)} - \psi_N(u_N)\right)$$
$$+ \operatorname{Re}\int_{\Gamma_1} \frac{\partial u_N}{\partial n}(a\bar{h}_N + \bar{h}_{Nt})\, d\Gamma + \frac{ap}{p+2}\|J_N u_N\|^{p+2}_{L_{p+2}(\Omega)}. \tag{3.8}$$

For the proof of (3.7) and (3.8), we adapt the proof of the corresponding identities in the defocusing case [11] to the focusing case.

From (3.7) and (3.8), and the fact that

$$\psi(J_N u_N) \leqslant \psi_N(u_N) \leqslant \psi(u_N),$$

we deduce the estimates

$$\|u_N\|^2_{L_2(\Omega)} \leqslant \|u_{N0}\|^2_{L_2(\Omega)} + 2\|h_N\|_{H^1(\Sigma_1)}\lambda_N, \tag{3.9}$$

$$\|\nabla u_N\|^2_{L_2(\Omega)} \leqslant \left|E(u_{N0})\right| + C\|u_N\|^{p+2}_{L_{p+2}(\Omega)} + C\lambda_N\|h_N\|_{H^1(\Sigma_1)} + C\int_0^t \|u_N\|^{p+2}_{L_{p+2}(\Omega)}\,ds, \tag{3.10}$$

where C depends on a and p,

$$\lambda_N(t) = \left(\int_0^t \left\|\frac{\partial u_N}{\partial n}\right\|^2_{L_2(\Gamma)} ds\right)^{\frac{1}{2}}, \tag{3.11}$$

and

$$E(\varphi) = \|\nabla\varphi\|^2_{L_2(\Omega)} - \frac{2}{p+2}\|\varphi\|^{p+2}_{L_{p+2}(\Omega)}, \quad \varphi \in H^1_{0,\Gamma_0}(\Omega).$$

We also recall the following lemmas (adapted to the focusing NLS), which will be very useful in our analysis.

Lemma 3.2. *(See [11].) Let u_N be a solution of the problem* (3.6) *and $q \in [C^1(\overline{\Omega})]^n$ be a real vector field with the property $q|_\Gamma = \hat{n}$, and let $H(x)$ be the $n \times n$ matrix with entries $H_{ij} = \frac{\partial q_i}{\partial x_j}$. Then, the following identity holds true.*

$$\begin{aligned}\frac{d}{dt}\operatorname{Im}\int_\Omega u_N\nabla\bar{u}_N\cdot q\,dx = {} & \operatorname{Im}\int_{\Gamma_1} h_N\bar{h}_{Nt}\,d\Gamma + 2\operatorname{Re}\int_\Omega H\nabla u_N\cdot\nabla\bar{u}_N\,dx \\ & + \|\nabla_A h_N\|^2_{\Gamma_1} - \left\|\frac{\partial u_N}{\partial n}\right\|^2_\Gamma + \operatorname{Re}\int_\Omega \nabla(\operatorname{div} q)\cdot\nabla u_N\bar{u}_N\,dx - \operatorname{Re}\int_{\Gamma_1}\operatorname{div} q\frac{\partial u_N}{\partial n}\bar{h}_N\,d\Gamma \\ & - \operatorname{Re}\int_\Omega \operatorname{div} q\mathcal{B}_N u_N\bar{u}_N\,dx - 2\operatorname{Re}\int_\Omega \mathcal{B}_N u_N\nabla\bar{u}_N\cdot q\,dx - 2\operatorname{Im}\int_\Omega a u_N\nabla\bar{u}_N\cdot q\,dx.\end{aligned} \tag{3.12}$$

Lemma 3.3. *(See [11].) Let u_N and q be as in Lemma* 3.2. *Then,*

$$-\operatorname{Re}\int_\Omega \operatorname{div} q\mathcal{B}_N u_N\bar{u}_N\,dx - 2\operatorname{Re}\int_\Omega \mathcal{B}_N u_N\nabla u_N\cdot q\,dx \leqslant M_1\|h_N\|^{p+2}_{L_{p+2}(\Gamma_1)} + M_2\psi_N(u_N) \tag{3.13}$$

where M_1, M_2 are positive constants.

Since u_N vanishes on a part of the boundary it satisfies the Poincaré inequality. That is, there exists a positive constant C which depends on Ω such that

$$\|u_N\|_{L_2(\Omega)} \leqslant C\|\nabla u_N\|_{L_2(\Omega)}. \tag{3.14}$$

From Lemmas 3.2, 3.3, and $\psi_N(u_N) \leqslant \psi(u_N)$, it follows that

$$\lambda_N^2 \leqslant C\|u_{N0}\|^2_{H^1_{0,\Gamma_0}(\Omega)} + C\|h_N\|^2_{H^1(\Sigma_1)} + C\|u_N\|^2_{H^1_{0,\Gamma_0}(\Omega)} + C\int_0^t \|u_N\|^2_{H^1_{0,\Gamma_0}(\Omega)}\,ds + C\int_0^t \|u_N\|^{p+2}_{p+2}\,ds, \tag{3.15}$$

where C depends also on q. Hence, we have

$$\lambda_N^2 \leqslant C + CD_N(t) + C\int_0^t D_N(s)\,ds + C\int_0^t \|u_N\|^{p+2}_{L_{p+2}(\Omega)}\,ds, \tag{3.16}$$

where

$$D_N(t)=\sup_{s\in[0,t]}\left\{\|u_N(s)\|^2_{H^1_{0,\Gamma_0}(\Omega)}\right\}$$

and C depends also on $\|u_{N0}\|_{H^1_{0,\Gamma_0}(\Omega)}$ and $\|h_N\|_{H^1(\Sigma_1)}$.

Now combining the Gagliardo–Nirenberg and Poincaré inequalities, we obtain

$$\|u_N\|^{p+2}_{L_{p+2}(\Omega)}\leqslant C\|\nabla u_N\|^{\theta(p+2)}_{L_2(\Omega)}\|u_N\|^{(1-\theta)(p+2)}_{L_2(\Omega)} \tag{3.17}$$

where

$$\frac{1}{p+2}=\frac{1}{2}-\frac{\theta}{n}.$$

Under our assumption $0<p<\frac{4}{n+2}$, we have

$$\theta(p+2)=\frac{np}{2}<2 \tag{3.18}$$

and

$$(1-\theta)(p+2)=p+2-\frac{np}{2}=\frac{2p+4-np}{2}.$$

Note that also

$$1\Big/\left(1-\frac{np}{4}\right)=\frac{4}{4-np}$$

and

$$\mu=\frac{2p+4-np}{2}\frac{4}{4-np}=\frac{4p}{4-np}+2<4. \tag{3.19}$$

Therefore, by Young's inequality, right-hand side of (3.17) is bounded by

$$\delta\|\nabla u_N\|^2_{L_2(\Omega)}+\frac{C}{\delta}\|u_N\|^{\mu}_{L_2(\Omega)}\leqslant\delta\|\nabla u_N\|^2_{L_2(\Omega)}+\delta\|u_N\|^4_{L_2(\Omega)}+C, \tag{3.20}$$

where C depends also on δ which is chosen to be sufficiently small. Hence, we have

$$\lambda_N^2\leqslant C+CD_N(t)+C\int_0^t D_N(s)\,ds+\delta\int_0^t\lambda_N^2\,ds \tag{3.21}$$

where C depends also on T.

By Gronwall's inequality,

$$\lambda_N^2\leqslant C+CD_N(t)+C\int_0^t D_N(s)\,ds.$$

Therefore,

$$D_N(t)\leqslant C+\delta\int_0^t D_N(s)\,ds+\delta\int_0^t\int_0^s D_N(\tau)\,d\tau\,ds$$

from which it follows that

$$D_N\leqslant C \tag{3.22}$$

where

$$C=C\big(\Omega,p,a,\|u_{N0}\|_{H^1_{0,\Gamma_0}(\Omega)},\|h_N\|_{H^1(\Sigma_1)},T,\delta\big).$$

4. Compactness

Using (3.22), and the convergence properties

$$\begin{cases} u_{N0} \to u_0, & \text{in } H^1_{0,\Gamma_0}(\Omega), \\ h_N \to h, & \text{in } H^1(\Sigma_1) \end{cases} \tag{4.1}$$

we conclude that

$$\begin{cases} \{u_N\} \text{ is a bounded sequence in } L_\infty\big(0,T;H^1_{0,\Gamma_0}(\Omega)\big), \\ \{J_N u_N\} \text{ is a bounded sequence in } L_{p+2}\big(0,T;L_{p+2}(\Omega)\big), \\ \big\{\mathcal{B}_N u_N = \mathcal{B} J_N u_N = |J_N u_N|^p J_N u_N\big\} \text{ is a bounded sequence in } L_\infty\big(0,T;L_{(p+2)'}(\Omega)\big). \end{cases} \tag{4.2}$$

By (4.2), it is evident that $\{u_N'\}$ is a bounded sequence in $L_\infty(0,T;H^{-1}(\Omega))$. Indeed,

$$\begin{aligned} \|u_N'\|_{H^{-1}(\Omega)} &= \sup_{\|\varphi\|_{H^1_0(\Omega)}=1} \big\{\big|(u_N',\varphi)\big|\big\} \\ &\leqslant \sup_{\|\varphi\|_{H^1_0(\Omega)}=1} \big\{\|\nabla u_N\|_{L_2(\Omega)}\|\varphi\|_{L_2(\Omega)} + \|J_N u_N\|^{p+2}_{L_{p+2}(\Omega)}\|\varphi\|_{L_{p+2}(\Omega)} + a\|u_N\|_{L_2(\Omega)}\|\varphi\|_{L_2(\Omega)}\big\} \leqslant C. \end{aligned} \tag{4.3}$$

It follows that $\{u_N\}$ has a subsequence (still denoted by $\{u_N\}$) such that

$$\begin{cases} u_N \to u & \text{weakly in } L_2\big(0,T;H^1_{0,\Gamma_0}(\Omega)\big), \\ \mathcal{B}_N u_N \to \chi & \text{weakly in } L_{(p+2)'}\big(0,T;L_{(p+2)'}(\Omega)\big), \\ u_N' \to u' & \text{weakly in } L_2\big(0,T;H^{-1}(\Omega)\big), \end{cases} \tag{4.4}$$

where χ is an element of $L_\infty(0,T;L_{(p+2)'}(\Omega))$.

Now, let's recall Aubin's compactness lemma.

Lemma 4.1. *(See [9].) The space*

$$\mathscr{X} = \big\{u \in L_2\big(0,T;H^1_{0,\Gamma_0}(\Omega)\big)\colon\ u' \in L_2\big(0,T;H^{-1}(\Omega)\big)\big\}$$

is compactly imbedded in $L_2(0,T;L_2(\Omega))$.

By Lemma 4.1, there exist a $u \in L_2(0,T;H^1_{0,\Gamma_0}(\Omega))$ and a subsequence of $\{u_N\}$ (still denoted by $\{u_N\}$) such that

$$\begin{cases} u_N \to u & \text{strongly in } L_2\big(0,T;L_2(\Omega)\big), \\ u_N \to u & \text{a.e. in } Q. \end{cases} \tag{4.5}$$

It follows that

$$\begin{cases} J_N u_N \to u & \text{a.e. in } Q, \\ \mathcal{B}_N u_N \to |u|^p u & \text{a.e. in } Q. \end{cases} \tag{4.6}$$

Now, we recall another well-known lemma.

Lemma 4.2. *(See [9].) Let Q be a bounded subset of $\mathbb{R}^n \times \mathbb{R}$, and let $\{g_N\}$ and g be elements $L_2(Q)$, $q \in (1,\infty)$, such that*

$$\|g_N\|_{L_q(Q)} \leqslant C \quad \textit{and} \quad g_N \to g \quad \textit{a.e. in } Q.$$

Then

$$g_N \to g \quad \textit{weakly in } L_q(Q).$$

By virtue of Lemma 4.2, it follows that $\chi = |u|^p u$ and

$$\mathcal{B}_N u_N \to |u|^p u \quad \text{weakly in } L_{(p+2)'}(Q) \equiv L_{(p+2)'}\big(0,T;L_{(p+2)'}(\Omega)\big).$$

The arguments up to this point proves Theorem 1.1.

5. Exponential decay

The proof of the exponential decay of solutions slightly differs than the proof of the existence theorem since we have to take into account the influence of the damping term and decay condition on the boundary control.

We begin with classical multipliers. In what follows, C will be a generic constant which might depend on Ω, p, a, $\|u_0\|_{H^1_{0,\Gamma_0}(\Omega)}$, M, b, but not on T.

We first multiply (1.1) by $\bar{u}$, integrate, and take imaginary parts to get

$$\frac{d}{dt}\|u\|^2_{L_2(\Omega)} = -2a\|u\|^2_{L_2(\Omega)} + 2\,\mathrm{Im}\int_{\Gamma_1}\frac{\partial u}{\partial n}\bar{h}\,d\Gamma \leqslant -2a\|u\|^2_{L_2(\Omega)} + 2\|h\|_{L_2(\Gamma_1)}\left\|\frac{\partial u}{\partial n}\right\|_{L_2(\Gamma)}. \tag{5.1}$$

Multiplying (5.1) by e^{2as} and integrating over $(0,t)$, we get

$$\|u\|^2_{L_2(\Omega)} \leqslant \|u_0\|^2 e^{-2at} + 2e^{-2at}\int_0^t e^{2as}\|h\|_{L_2(\Gamma_1)}\left\|\frac{\partial u}{\partial n}\right\|_{L_2(\Gamma)} ds. \tag{5.2}$$

Secondly, we multiply (1.1) by $\bar{u}_t$, integrate, take real parts, and obtain

$$\frac{d}{dt}E(u) = -2aE(u) + 2\,\mathrm{Re}\int_{\Gamma_1}\frac{\partial u}{\partial n}(a\bar{h}+\bar{h}_t)\,d\Gamma + \frac{2ap}{p+2}\|u\|^{p+2}_{L_{p+2}(\Omega)}. \tag{5.3}$$

Multiplying (5.3) by e^{2as} and integrating over $(0,t)$, we have

$$\begin{aligned}\|\nabla u\|^2_{L_2(\Omega)}e^{2at} &= E(u_0) + \frac{2}{p+2}\|u\|^{p+2}_{L_{p+2}(\Omega)}e^{2at} \\ &\quad + 2\,\mathrm{Re}\int_0^t e^{2as}\int_{\Gamma_1}\frac{\partial u}{\partial n}(a\bar{h}+\bar{h}_t)\,d\Gamma\,ds + \frac{2ap}{p+2}\int_0^t e^{2as}\|u\|^{p+2}_{L_{p+2}(\Omega)}ds. \end{aligned}\tag{5.4}$$

(5.1) and (5.3) show that conservation laws for mass and energy do not hold as opposed to the homogeneous equation.

Lemma 5.1. *Let q be a sufficiently smooth vector field on $\overline{\Omega}$ such that $q = \hat{n}$ on Γ. Let also H be the $n\times n$ matrix with entries $H_{ij} = \frac{\partial q_i}{\partial_j}$. Then,*

$$\begin{aligned}\frac{d}{dt}\mathrm{Im}\int_\Omega u\nabla\bar{u}\cdot q\,dx &= \mathrm{Im}\int_{\Gamma_1}h\bar{h}_t\,d\Gamma + 2\,\mathrm{Re}\int_\Omega H\nabla u\cdot\nabla\bar{u}\,dx + \|\nabla_A h\|^2_{L_2(\Gamma_1)} - \left\|\frac{\partial u}{\partial n}\right\|^2_{L_2(\Gamma_1)} \\ &\quad + \mathrm{Re}\int_\Omega \nabla(\mathrm{div}\,q)\cdot\nabla u\bar{u}\,dx - \mathrm{Re}\int_{\Gamma_1}\frac{\partial u}{\partial n}\bar{h}\,\mathrm{div}\,q\,d\Gamma + \frac{2}{p+2}\|h\|^{p+2}_{L_{p+2}(\Gamma_1)} \\ &\quad + \frac{p}{p+2}\int_\Omega(\mathrm{div}\,q)|u|^{p+2}\,dx - 2a\,\mathrm{Im}\int_\Omega u\nabla\bar{u}\cdot q\,dx. \end{aligned}\tag{5.5}$$

Proof. The proof is omitted since it is very similar to the proof of Lemma 3.2. □

From Lemma 5.1, it follows that

$$\begin{aligned}\int_0^t\left\|\frac{\partial u}{\partial n}\right\|^2_{L_2(\Gamma_1)}e^{2as}\,dt &\leqslant Ce^{2at}\|\nabla u\|^2_{L_2(\Omega)} + C\|\nabla u_0\|^2_{L_2(\Omega)} + C\int_0^t e^{2as}\|h\|^2_w\,ds \\ &\quad + C\int_0^t e^{2as}\|\nabla u\|^2_{L_2(\Omega)}ds + C\int_0^t e^{2as}\|u\|^{p+2}_{L_{p+2}(\Omega)}ds. \end{aligned}\tag{5.6}$$

By the Gagliardo–Nirenberg and Poincaré inequalities,

$$\|u\|^{p+2}_{L_{p+2}(\Omega)} \leqslant C\|\nabla u\|^{\theta(p+2)}_{L_2(\Omega)}\|u\|^{(1-\theta)(p+2)}_{L_2(\Omega)} \leqslant \delta\|\nabla u\|^2_{L_2(\Omega)} + \frac{C}{\delta}\|u\|^{\mu}_{L_2(\Omega)} \tag{5.7}$$

where θ and μ are as in (3.18) and (3.19), respectively.

By (5.2),

$$\begin{aligned}\|u\|_{L_2(\Omega)}^{\mu} &\leqslant \left(\|u_0\|_{L_2(\Omega)}^2 e^{-2at} + 2e^{-2at}\int_0^t e^{2as}\|h\|_{L_2(\Gamma_1)}\left\|\frac{\partial u}{\partial n}\right\|_{L_2(\Gamma)} ds\right)^{\frac{\mu}{2}} \\ &\leqslant Ce^{-\mu at}\left(\|u_0\|_{L_2(\Omega)}^{\mu} + \left(\int_0^t e^{2as}\|h\|_{L_2(\Gamma_1)}^2 ds\right)^{\frac{\mu}{4}}\left(\int_0^t e^{2as}\left\|\frac{\partial u}{\partial n}\right\|_{L_2(\Gamma)}^2 ds\right)^{\frac{\mu}{4}}\right) \\ &\leqslant Ce^{-\mu at}\big(1+\delta H^2(t)\lambda^2(t)\big) \end{aligned} \tag{5.8}$$

where

$$H(t) = \left(\int_0^t e^{2as}\|h\|_{L_2(\Gamma_1)}^2 ds\right)^{\frac{1}{2}} \quad \text{and} \quad \lambda(t) = \left(\int_0^t e^{2as}\left\|\frac{\partial u}{\partial n}\right\|_{L_2(\Gamma)}^2 ds\right)^{\frac{1}{2}}. \tag{5.9}$$

Therefore,

$$\int_0^t e^{2as}\|u\|_{L_{p+2}(\Omega)}^{p+2} ds \leqslant C + \delta\int_0^t e^{2as}\|\nabla u\|_{L_2(\Omega)}^2 ds + \delta\int_0^t e^{(2-\mu)as}H^2(s)\lambda^2(s)\,ds. \tag{5.10}$$

Now, by (5.6),

$$\lambda^2(t) \leqslant \overbrace{C + CD(t) + C\int_0^t D(s)\,ds}^{\alpha(t)} + \int_0^t \overbrace{\delta e^{(2-\mu)as}H^2(s)}^{\beta(s)}\lambda_N^2(s)\,ds, \tag{5.11}$$

where we set

$$D(t) = \sup_{s\in[0,t]}\left\{\left\|\nabla u(s)\right\|_{L_2(\Omega)}^2 e^{2as}\right\}.$$

Then by Gronwall's inequality and condition (1.8), it follows that

$$\lambda^2 \leqslant \left(C + CD(t) + C\int_0^t D(s)\,ds\right)\exp\left(\delta\int_0^t e^{(2-\mu)as}H^2(s)\,ds\right) \leqslant \left(C + CD(t) + C\int_0^t D(s)\,ds\right). \tag{5.12}$$

By (5.4),

$$D(t) \leqslant C + \delta\int_0^t D(s)\,ds + \int_0^t \beta(s)\lambda^2(s)\,ds. \tag{5.13}$$

Hence,

$$D(t) \leqslant C + \delta\int_0^t D(s)\,ds + \delta\int_0^t\int_0^s D(\tau)\,d\tau\,ds. \tag{5.14}$$

Adding $\delta\int_0^t D(s)\,ds$ to both sides we have

$$\tilde{D}(t) = D(t) + \delta\int_0^t D(s)\,ds \leqslant C + 2\delta\int_0^t \tilde{D}(s)\,ds$$

which implies

$$\tilde{D}(t) \leqslant Ce^{2\delta t}.$$

In particular

$$D(t) \leqslant e^{2\delta t}$$

and

$$\|\nabla u\|^2_{L_2(\Omega)} \leqslant Ce^{2(\delta-a)t}.$$

We notice that C does not depend on T. Hence, we have Theorem 1.2.

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